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PROPERTIES OF BATCH MEANS FROM
STATIONARY ARMA TIME SERIES

KEEBOM KANG
BRUCE SCHMEISER

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12 PERSONAL AUTHOR(S)

Kang, Keebom (University of Miami) and Schmeiser, Bruce W.

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The batch means process arising from an arbitrary autoregressive moving-average (ARMA) process time series is derived. As side results, the variance and correlation structures of the batch means process as functions of the batch size and parameters of the original process are obtained. Except for the first-order ARMA process, for which a closed-form expression is obtained, the parameters of the batch-means process are determined numerically.

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1. INTRODUCTION

Corresponding to any time series $\{X_i ; i = 0,1,2, \cdots \}$ is the time series $\{\bar{X}_j ; j = 0,1,2, \cdots \}$, where

$$\bar{X}_j = b^{-1} \sum_{h=1}^{b} X_{(j-1)b+h} \tag{1}$$

is the j^{th} batch mean of size b. The batch-means process, or aggregated time series, is of interest when observations are actually batched (see, e.g., Telser [1967], Amemiya and Wu [1972], and Tiao [1972]), when calculations need to be simplified (see, e.g., Blackman and Tukey [1958, Sec. B. 17]), or when the process mean, $E(X)$, is to be estimated. The third context motivates our work.

Consider estimating $E(X)$ with the average of n observations, $\bar{X} = \sum_{i=1}^{n} X_i / n$. Using batch means to estimate the variance of the sample mean, $V(\bar{X})$, has long been considered (see, e.g., Conway [1963]). Brillinger (1973) shows that if the values of a process at a distance from each other are only weakly dependent, then the batch means are asymptotically independent and normally distributed. Thus, for large batch size b, $V(\bar{X})$ can be estimated using S^2_k / k, where $k = \lfloor n / b \rfloor$, $S^2_k = (k-1)^{-1} \left[\sum_{j=1}^{k} \bar{X}_j - k\bar{X}^2 \right]$, and $\lfloor \cdot \rfloor$ denotes the floor function. (Moran [1965] discusses related estimators.) A nominal $100(1-\alpha)$ percent confidence interval on $E(X)$ is then $\bar{X} \pm t_{1-\alpha/2,k-1} S_k k^{-1/2}$, where $t_{1-\alpha/2,k-1}$ is the $1-(\alpha/2)$ quantile of Student's t distribution with $k-1$ degrees of freedom.

The batch means algorithms developed by Mechanic and McKay (1966), Fishman (1978), Law and Carson (1979), Schriber and Andrews (1979), and Adam (1983) empirically calculate measures of batch dependency for various batch sizes b in an attempt to determine a reasonably small value of b that yields batch means that are almost independent and normally distributed. These procedures require substantial calculation; Law and Carson (1979), for example, calculate first-order correlations based on 400 batches. That so many batches are required for accurate estimation of dependency measures is unfortunate, since Schmeiser (1982) shows that, for fixed n, additional batches beyond some small number (ten to thirty) do little to improve the statistical properties of the batch means confidence interval procedures. The results of Section 2 are motivated by the idea that knowledge of the relationship between $\{X_i\}$ and $\{\bar{X}_j\}$ can be used to measure properties of $\{\bar{X}_j\}$ even for small values of k.

A second reason for studying the relationship between $\{X_i\}$ and $\{\bar{X}_j\}$ is to allow more efficient simulation studies of batch-means procedures. Studying the performance of several batch-means procedures in the context of various distributions assumptions for $\{X_i\}$ requires a large computational effort, especially when the large sample sizes required to simulate a system and the large number of replications required for meaningful conclusions are considered. A crude Monte Carlo method is to generate $X_1, X_2, \cdots, X_n$ and calculate the batch means $\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_{\lfloor n/b \rfloor}$ for all values of b of interest. A computationally more efficient alternative is to derive the properties of $\{\bar{X}_j\}$ from the properties of $\{X_i\}$ and to generate directly the batch means $\{\bar{X}_j\}$, as discussed in Section 4.

A third motivation is that direct insights might result from studying properties of $\{\bar{X}_j\}$ as functions of the parameters of $\{X_i\}$. Particularly interesting is the sensitivity of $\{\bar{X}_j\}$ to the underlying process and to the batch size b, as discussed in Kang (1984, Chapter 5).

Section 2 contains results relating batch-means processes to arbitrary stationary autoregressive moving-average (ARMA) processes. Section 3 considers the special case of the underlying process being ARMA (1,1). Section 4 is a summary containing an algorithm for determining the batch-means process from the underlying process.

2. BATCH MEANS OF STATIONARY ARMA PROCESSES

The ARMA (p,q) process $\{X_i\}$ by definition satisfies

$$\sum_{h=0}^{p} \phi_h X_{i-h} = \sum_{h=0}^{q} \theta_h \epsilon_{i-h} \tag{2}$$

where $\phi_0 \equiv 1$, $\theta_0 \equiv 1$, and the error terms ϵ_i are independent with zero mean and variance σ_ϵ^2. The main result of this paper is Theorem 1, which states that batch-means processes arising from ARMA underlying processes are themselves ARMA and specifies the parameter values.

Theorem 1. *Consider the stationary ARMA(p,q) process of equation (2). Then $\{\bar{X}_j\}$ is the stat onary ARMA$(p,\bar{q})$ process*

$$\sum_{h=0}^{p} \bar{\phi}_h \bar{X}_{j-h} = \sum_{h=0}^{\bar{q}} \bar{\theta}_h \bar{\epsilon}_{j-h}$$

where $\bar{\phi}_0 \equiv 1$, $\bar{\theta}_0 \equiv 1$, the batch-means error terms $\bar{\epsilon}_j$ are independent random variables with zero mean and variance $\sigma^2_{\bar{\epsilon}}$, and $\bar{q}$, $\bar{\phi}_1, \bar{\phi}_2, \cdots, \bar{\phi}_p$, and $\bar{\theta}_1, \bar{\theta}_2, \cdots, \bar{\theta}_{\bar{q}}$ are functions (of the parameters of the underlying process and the batch size b) given in Lemmas 1, 2, and 4, respectively.

The proof of Theorem 1 requires the following lemmas.

Lemma 1 (Anderson [1979a, p. 155]). *If the underlying process $\{X_i\}$ is a stationary ARMA(p,q) process, then the batch-means process $\{\bar{X}_j\}$ is a stationary ARMA$(p,\bar{q})$ process, where $\bar{q} = p - \lfloor (p-q)/b \rfloor$.*

Anderson uses the more complicated, but equivalent, expression $\bar{q} = \left\lfloor \frac{q+(p+1)(b-1)}{b} \right\rfloor$. Lemma 1 has several direct implications, as discussed in the Appendix.

Lemma 2 (Amemiya and Wu [1972]). *The AR parameters $\bar{\phi}_1, \bar{\phi}_2, \cdots, \bar{\phi}_p$ of the batch means process $\{\bar{X}_j\}$ are the coefficients of $B^1, B^2, \cdots, B^p$ of $\prod_{h=1}^{p}(1-\alpha_h^{\,b} B)$, respectively, where $\alpha_1, \alpha_2, \cdots, \alpha_p$ are the roots of the characteristic equation $\Phi(B) \equiv \sum_{h=0}^{p} \phi_h B^{p-h} = 0$.*

Lemma 3. *For any stationary process $\{X_i\}$, the lag-h autocorrelation of the batch-means process $\{\bar{X}_j\}$ is*

$$\bar{\rho}_h \equiv \mathrm{Corr}(\bar{X}_j, \bar{X}_{j+h}) = \left[\sum_{i=1}^{b} i\,\rho_{(h-1)b+i} + \sum_{i=1}^{b-1} i\,\rho_{(h-1)b+2b-i}\right] / [bc]$$

where $c = 1 + 2\sum_{h=1}^{b-1}(1-(h/b))\rho_h$, $\rho_h \equiv R_h / R_0$, *and* $R_h \equiv \mathrm{E}[(X_i - \mathrm{E}\,X)(X_{i+h} - \mathrm{E}\,X)]$ *for* $h = 0,1,2,\cdots$ *and* $i = 0,1,2,\cdots$.

Proof. For any stationary process, the lag-h covariance of the batch means process is

$$\mathrm{Cov}(\bar{X}_j, \bar{X}_{j+h}) = b^{-2}\sum_{i=1}^{b}\sum_{k=1}^{b}\mathrm{Cov}(X_{(j-1)b+i}, X_{(j+h-1)b+k}) \tag{3}$$

(see, e.g., Kleijnen [1975, p. 507]), which is a special case of the covariance of linear combinations of random variables as discussed in Box and Jenkins (1976, pp. 28-29). Also, for any stationary process, each batch mean $\bar{X}_j$ has variance

$$\bar{R}_0 \equiv \mathrm{V}(\bar{X}_j) = c\,R_0 / b \tag{4}$$

(see, e.g., Fishman [1973, p. 281]). The definition of correlation

$$\bar{\rho}_h \equiv \mathrm{Cov}(\bar{X}_j, \bar{X}_{j+h}) / [\mathrm{V}(\bar{X}_j)\mathrm{V}(\bar{X}_{j+h})]^{1/2}$$

and equations (3) and (4) yield

$$\bar{\rho}_h = b^{-2}\sum_{i=1}^{b}\sum_{k=1}^{b}\mathrm{Cov}[X_{(j-1)b+i}, X_{(j+h-1)b+k}] / [c\,R_0 / b]$$

$$= \sum_{i=1}^{b}\sum_{k=1}^{b}\mathrm{Corr}[X_{(j-1)b+i}, X_{(j+h-1)b+k}] / [bc].$$

Counting like terms arising from stationarity yields the result. □

Lemma 4 (Anderson [1971, p. 237]). *Consider a stationary ARMA(p,$\bar{q}$) process with known AR (autoregressive) parameters $\bar{\phi}_1, \bar{\phi}_2, \cdots, \bar{\phi}_p$; variance $\bar{R}_0$, and autocorrelation coefficients $\bar{\rho}_1, \bar{\rho}_2, \cdots, \bar{\rho}_{\bar{q}}$. Then the MA (moving-average) parameters $\bar{\theta}_1, \bar{\theta}_2, \cdots, \bar{\theta}_{\bar{q}}$ are determined.*

Theorem 1 can now be proven using the four lemmas.

Proof of Theorem 1.

1. From Lemma 1, the AR and MA orders of $\{\bar{X}_j\}$ are determined: in particular, $\{\bar{X}_j\}$ is an ARMA(p,$\bar{q}$) process.
2. From Lemma 2, the AR parameters of $\{\bar{X}_j\}$ are determined.
3. The autocorrelations $\rho_1, \rho_2, \cdots, \rho_{b(\bar{q}+1)-1}$ of a stationary ARMA process can be calculated using the algorithm of Sweet and Mazaheri (1979).

4. Given the autocorrelations from Step 3, the batch-means variance $\bar{R}_0$ is determined by equation (4) and the batch-means autocorrelations $\bar{\rho}_1, \bar{\rho}_2, \cdots, \bar{\rho}_{\bar{q}}$ are determined by Lemma 3.

5. The MA parameters $\bar{\theta}_1, \bar{\theta}_2, \cdots, \bar{\theta}_{\bar{q}}$ are then determined from Lemma 4. □

The representation of the batch-means process is not unique. The batch-means MA parameters of Lemma 4 require the solution of a polynomial equation of order $2\bar{q}$ to determine $\bar{\theta}_1, \bar{\theta}_2, \cdots, \bar{\theta}_{\bar{q}}$. The $2\bar{q}$ roots can be partitioned $2^{\bar{q}}$ ways into two sets $(x_1, x_2, \cdots, x_{\bar{q}})$ and $(x_{\bar{q}+1}, x_{\bar{q}+2}, \cdots, x_{2\bar{q}})$ such that $x_{\bar{q}+i} = 1/x_i$. All such subsets of size $\bar{q}$ from $(x_1, x_2, \cdots, x_{2\bar{q}})$ determine $\bar{\theta}_1, \bar{\theta}_2, \cdots, \bar{\theta}_{\bar{q}}$ corresponding to stochastically equivalent processes. But there is a unique subset of size $\bar{q}$ having $|x_i| \leqslant 1$, and therefore $|x_{\bar{q}+i}| = |1/x_i| \geqslant 1$, for $i = 1, 2, \cdots, \bar{q}$. Thus for the ARMA$(p, \bar{q})$ process there exist $2^{\bar{q}} - 1$ non-invertible processes corresponding to a unique invertible process.

3. ARMA(1,1) BATCH MEANS

Now consider the special case of stationary first-order ARMA processes

$$X_i + \phi X_{i-1} = \epsilon_i + \theta \epsilon_{i-1} \qquad \text{for } i = 1, 2, \cdots \tag{5}$$

The low-order moments of $\{X_i\}$ are the zero mean, variance

$$R_0 = \sigma_\epsilon^2 (1 + \theta^2 - 2\phi\theta) \,/\, (1 - \phi^2)\,, \tag{6}$$

the lag-one autocorrelation

$$\rho_1 = (1 - \phi\theta)(\theta - \phi) \,/\, (1 + \theta^2 - 2\phi\theta)\,. \tag{7}$$

and lag-h autocorrelations

$$\rho_h = (-\phi)\rho_{h-1} = (-\phi)^{h-1}\rho_1 \qquad \text{for } h = 2, 3, \cdots \tag{8}$$

Closed-form expressions for the parameters of the batch-means process are given in Theorem 2.

Theorem 2. *Consider the stationary ARMA(1,1) process of equation (5). The corresponding batch-means process is the stationary ARMA(1,1) process*

$$\bar{X}_j + \bar{\phi} \bar{X}_{j-1} = \bar{\epsilon}_j + \bar{\theta} \bar{\epsilon}_{j-1} \qquad \text{for } j = 1, 2, \cdots \tag{9}$$

where

$$\bar{\phi} = (-1)^{b+1} \phi^b \tag{10}$$

$$\bar{\theta} = \frac{(2\bar{\phi}\bar{\rho}_1 + \bar{\phi}^2 + 1) \pm [(1 - \bar{\phi}^2)(1 - (2\bar{\rho}_1 + \bar{\phi})^2)]^{1/2}}{2(\rho_1 + \bar{\phi})} \tag{11}$$

and

$$\sigma_{\bar{\epsilon}}^2 = \bar{R}_0\,(1-\bar{\phi}^2)\ /\ (\bar{\theta}^2 - 2\bar{\phi}\bar{\theta} + 1) \tag{12}$$

where

$$\bar{R}_0 = c\ R_0\ /\ b \tag{13}$$

$$\bar{\rho}_1 = \rho_1(1+\bar{\phi})^2\ /\ [bc\,(1+\phi)^2] \tag{14}$$

and

$$c\ =\ 1 + \{2\rho_1[(-\phi)^b\ +\ b\,(1+\phi) - 1]\ /\ [b\,(1+\phi)^2]\} \tag{15}$$

Proof of Theorem 2.

Equations (13), (14), and (15) follow from Lemma 3 and equation (4) via equation (8). Since $\{X_i\}$ is ARMA(1,1), $\{\bar{X}_j\}$ is also ARMA(1,1) by Lemma 1: that is, equation (9) holds. Equation (10) is a special case of Lemma 2. Since $\{\bar{X}_j\}$ is ARMA(1,1), equation (7) yields

$$\bar{\rho}_1 = (1-\bar{\phi}\bar{\theta})(\bar{\theta}-\bar{\phi})\ /\ (1+\bar{\theta}^2 - 2\bar{\phi}\bar{\theta})\ .$$

Solving for $\bar{\theta}$ yields the two roots of equation (11). Since $\{\bar{X}_j\}$ is an ARMA(1,1) process, equation (6) holds with the batch-means parameters:

$$\bar{R}_0 = \sigma_{\bar{\epsilon}}^2(1+\bar{\theta}^2 - 2\bar{\phi}\bar{\theta})\ /\ (1-\bar{\phi}^2)\ .$$

Solving for the variance of the batch-means error term, $\sigma_{\bar{\epsilon}}^2$, with either value of $\bar{\theta}$ from equation (11) yields equation (12). (But note that the value of $\sigma_{\bar{\epsilon}}^2$ depends on the choice of $\bar{\theta}$). □

4. SUMMARY

Properties of batch means are studied under the assumption that the underlying process is ARMA(p,q). For ARMA(1,1) processes, closed-form expressions for the corresponding batch-means processes are obtained. A numerical procedure is developed for calculating the parameters of the ARMA batch-means process from the parameters of the underlying process and the batch size b. This procedure is stated concisely here for convenience.

ARMA(p,q) Procedure.

Given parameters $\phi_1,\phi_2,\cdots,\phi_p$, $\theta_1,\theta_2,\cdots,\theta_q$, error variance σ_ϵ^2, and batch size b, calculate

1. $\bar{q} = p - \lfloor (p-q)/\ b \rfloor$
2. $\bar{\phi}_1,\bar{\phi}_2,\cdots,\bar{\phi}_p$ using Lemma 2
3. $R_0,\rho_1,\rho_2,\cdots,\rho_{b(\bar{q}+1)-1}$ from the Yule-Walker equations, probably using the algorithm of Sweet and Mazaheri (1979)
4. $c = 1 + 2\sum_{h=1}^{b-1}(1-(h\ /\ b))\rho_h$

5. $\bar{R}_0$ from equation (4)

6. $\bar{\rho}_1, \bar{\rho}_2, \cdots, \bar{\rho}_{\bar{q}}$ using Lemma 3

7. $\bar{\theta}_1, \bar{\theta}_2, \cdots, \bar{\theta}_{\bar{q}}$ from Lemma 4

8. $\sigma_{\bar{\epsilon}}^2$ from equation (13) of Anderson (1971, p. 237).

A FORTRAN implementation of the ARMA(p, q) procedure is given in Kang (1984).

When the underlying process is ARMA(1,1), the following closed-form procedure can be used:

ARMA(1,1) Procedure.

Given parameters ϕ, θ, error variance σ_ϵ^2, and batch size b, calculate

1. $\bar{q} = 1$
2. $\bar{\phi}$ from equation (10)
3. R_0 from equation (6), ρ_1 from equation (7)
4. c from equation (15)
5. $\bar{R}_0$ from equation (13)
6. $\bar{\rho}_1$ from equation (14)
7. $\bar{\theta}$ from equation (11)
8. $\sigma_{\bar{\epsilon}}^2$ from equation (12)

Notice in the ARMA(1,1) special case that calculation in step 3 of all $2b-1$ autocorrelations of the underlying process is not necessary.

If the underlying error terms ϵ_i are normally distributed, then the batch-means error terms $\bar{\epsilon}_j$ are also normally distributed (see, e.g., Johnson and Kotz [1971, p. 51]). Therefore, generation of random variates directly from the batch-means process is straightforward using equation (9), thereby avoiding the costly computations of aggregating observations from the underlying process. Initialization for steady-state results is straightforward for AR and MA processes, but initialization for ARMA is complicated unless the process is warmed-up by discarding some initial observations (Anderson [1979b]).

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APPENDIX

Although Lemma 1 is simple to state compactly, its implications are more clear when cases are considered individually:

If	and	then
$p > q$	$b < p - q$	$q \leqslant \bar{q} \leqslant p - 1$
	$b = p - q$	$\bar{q} = p - 1$
	$b > p - q$	$\bar{q} = p$
$p = q$		$\bar{q} = p$
$p < q$	$b < q - p$	$p + 2 \leqslant \bar{q} \leqslant q$
	$b = q - p$	$\bar{q} = p + 1$
	$b > q - p$	$\bar{q} = p + 1$

Many results can be stated immediately from examination of these individual cases. Five such results are:

Result 1. *If* $\{X_i\}$ *is* $AR(p)$*, then* $\{\bar{X}_j\}$ *is* $ARMA(p, \bar{q})$*, as studied by Amemiya and Wu (1972). Additionally,* $1 \leqslant \bar{q} \leqslant p$.

Result 2. *If* $\{X_i\}$ *is* $MA(q)$*, then* $\{\bar{X}_j\}$ *is* $MA(\bar{q})$*, where* $1 \leqslant \bar{q} \leqslant q$.

Result 3. *If* $\{X_i\}$ *is AR or ARMA with batch size satisfying* $0 \leqslant p - q < b$*, then* $\{\bar{X}_j\}$ *is* $ARMA(p, p)$.

Result 4. *If* $p \geqslant q$*, then* $\lim_{b \to \infty} \bar{q} = p$.

Result 5. *If* $p < q$*, then* $\lim_{b \to \infty} \bar{q} = p + 1$.

Of course, considering only the order of the batch-means process can be misleading. For example, Results 4 and 5 indicate that large batches lead to MA components of order p or $p+1$; in particular, a batched MA(q) process converges to an MA(1) process. But large batches are asymptotically independent. The explanation is that $\bar{\theta}_1$ is approaching zero as batch size increases. An implication is that, even for this nicest case of ARMA underlying processes, estimation of the order of the batch-means process is likely to be difficult.

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